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The Theory of Search*

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1. INTRODUCTION

The description of economic behavior as sequential decision making under uncertainty has been hampered by the limited nature of the available theoretical results. This paper tries to fill some of the gaps. In particular, the paper develops some basic results for the problem of search.

We present a general formulation of the search problem in Section 2. Search is seen as sequential sampling from a population X. The sample points x_t could be prices in different stores for a given good, the qualities of nonhomogeneous goods, job offers, or bids for an asset, to name several obvious examples; for concreteness, we occasionally refer to a searching consumer. Our results apply to the general search problem and in no way depend on any given interpretation or example.

An individual engaged in search is assumed to have a probability distribution over X. This might be an "objective" distribution known with certainty or an uncertain prior that will be revised in the light of additional sample information. We show that in very general circumstances the optimal decision rule of an expected utility maximizer takes the form of a *switchpoint* level of utility s. If the utility of the best x_t available so far is higher than s, the search will end; otherwise it will continue. As new samples are drawn, s may change.

The switchpoint can in general be characterized only in terms of a functional equation that is impossible to solve explicitly. Therefore, in Section 3 we develop calculable and easily interpretable upper and lower bounds for the switchpoint. In so doing we show that the switchpoint when the distribution is not known with certainty is at least as great as that when the distribution is known with certainty. This is because in the former case the potential benefits of continued search include the acquisition of additional information about the distribution. In most real situa-

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tions the probability distribution will not be known with certainty, and we will observe adaptive or learning behavior. Only when priors are very strong relative to new sample information will a model with no learning about the distribution provide a good approximation to the truth.

In Section 4 we discuss the effect on the switchpoint—or, equivalently, on the duration of search—of changes in the rate of time preference, in risk aversion, in search costs, and in characteristics of the distribution. As expected, the switchpoint level of utility falls with an increase in the rate of time preference or an increase in risk aversion. The result of both changes is to make costs today weigh more heavily against future benefits and to reduce the incentive to search. Also, as expected, the switchpoint falls with an increase in search costs. Two types of changes in the probability distribution are considered, a translation and a change in dispersion or risk. A translation to the right raises the switchpoint. An increase in risk in the sense of Rothschild and Stiglitz [12] may either raise or lower the switchpoint. An increase in risk in the sense of Diamond and Stiglitz [6] unambiguously raises the switchpoint and with it the value of the whole search procedure; this implies that even a risk averse individual will *prefer* to sample from a more risky distribution.

2. The Optimal Policy

A decisionmaker D is sampling sequentially from a population X. After drawing each observation x_t, possibly a vector, he must decide whether or not to continue sampling. At such a decision point he has a nondegenerate joint probability distribution F over hypothetical future infinite sequences of drawings from X. From F a marginal distribution F^n over the nth draw may be derived.

It should be emphasized that the distribution F incorporates D's beliefs about future observations and, possibly, a sampling strategy. For example, if D's opinion about x_{t+k} would be influenced by x_t, then F would exhibit a dependence of x_{t+k} on x_t; or if D's plan is to draw the *a priori* most promising samples first, then the marginals F^n would become less favorable as n increased.

We distinguish two generic types of search problem. The problem which arises when D is sampling from a mutually independent and identically distributed population with known distribution we call the *static* case; by definition, there is no learning in the static case. Any search problem not of this type is called *adaptive*. An important example of an adaptive problem is Bayesian learning about an unknown parameter of the distribution of a mutually independent and identically distributed (*iid*) population; here, unlike in the static case, D learns from sample information.

It is assumed that D evaluates a future stream of utility by its present value with respect to a positive rate of time preference r.

If D samples for the jth time and draws x_j, he enjoys an immediate utility "payoff" of $k_j(x_j)$, which may be random. A payoff function of the form $k_j(x_j) = -c(j)$, where $c(j)$ is a nonnegative and possibly random search cost,[1] we call the *inspection* case: the object x_j is observed but not consumed or enjoyed when it is sampled. We call a payoff function of the form $k_j(x_j) = u(x_j)$ the *experience* case: $u(x_j)$ is the utility derived from actually consuming or enjoying x_j for a single period and no search cost is incurred.[2] Of course, a combination of the two cases is possible,[3] and all our results are proved for this more general situation.

If D does not sample, he receives an immediate payoff y, which is the utility of the best previously observed x_t. Thus, if D has already sampled j times,

$$y = \max(u(x_1),..., u(x_j)). \tag{1}$$

The possibility of *recall*, which allows D to use any previously observed x_t, is obviously unrealistic in some situations. For example, recall is not available when a prospective employer will give the job to someone else if D does not immediately accept; however, recall is appropriate in the case of a searching consumer if store prices are fixed.

D may continue to sample as long as the wishes, and once he stops we assume that he is able to enjoy y forever. The time unit is the sampling period, which is taken to be constant.

We wish to stress that with only trivial changes in the proofs, *the results*

[1] For instance, $c(j)$ could be constant, increasing, or decreasing with j, or an iid random variable. Note that $c(j)$ is measured in utils. The fact that $c(j)$ is nonnegative is only a matter of convenience; it means that we have normalized the utility function so that zero utility corresponds to sampling free of charge.

[2] This terminology is not always appropriate. For example, if x_j is a price, then x_j is not really consumed and $u(\ )$ is not really the utility function (it is the indirect utility function). It should be mentioned that Nelson [9] distinguishes between search and experience goods. Search goods—for example, consumer durables—are observed during the search process but not enjoyed; only after the search process has ended and a purchase is made does the searcher begin to enjoy a stream of services for the good. On the other hand, experience goods are "observed" by actually consuming them: restaurant meals and new types of food fall into this category. In the first case search costs are usually the actual costs of physical search—transportation, time, etc. In the second, search costs consist of the disutility of consuming a "bad observation." Hirshleifer [7] has suggested, and we concur, that the first type of good might more usefully be called an inspection good.

[3] In this case $k_j(_j) \leqslant u(x_j)$.

of this paper hold as well if there is no possibility of recall or if D may sample no more than a fixed and finite number of times or if, after quitting, he is able to enjoy y for only a fixed and finite number of periods.[4]

D's problem is to choose a policy, d, which will tell him not only whether to sample now, but also whether to sample after having observed any conceivable sequence of realizations of the random variable x_t in the future. Formally, if $x = (x_1, x_2, x_3, ...)$ is an infinite sequence of possible future observations, let $d(x) = n \geqslant 0$ indicate that sampling would terminate immediately after the nth draw. For example, consider the possible sequence of observations (3, 1, 4, 1, 5, 9,...) and suppose that $d(3, 1, 4, 1, 5, 9, ...) = 4$. Then, if D follows the policy d, he will stop after observing (3, 1, 4, 1). It is clear that the decision to stop after having drawn a particular n observations can depend only on these observations and not on the values of any possible subsequent observations. Therefore consistency requires that if $d(x_1, ..., x_n, x_{n+1}, x_{n+2}, ...) = n$, then $d(x_1, ..., x_n, x'_{n+1}, x'_{n+2}, ...) = n$ for arbitrary $x'_{n+1}, x'_{n+2}, ...$. Hence, if ever $d(x) = 0$, then d must be identically zero. Note that $d(x) \equiv n$ corresponds to the policy "always take n samples and then stop."

We have not considered policies which allow D to resume sampling once he has stopped. This is referred to in the statistical literature as an optimal stopping problem, and the results to be obtained for this problem hold under quite general F and $\{k_j(\ )\}$. However, it is fair to ask under what condition would D never *wish* to resume sampling once he stopped. A sufficient condition is that (a) D's opinions about the future should be influenced (if at all) only by sample information and *not* by calendar time and that (b) $k_j(\ )$ does not depend on calendar time. Under (a) and (b), if the best action at some decision point is not to sample, then this must remain the best action as the context in which the decision is to be made will not have changed. The distribution over future samples does not change because by (a) it can be modified only in light of new sample information; the cost structure $\{k_j(\ )\}$ remains the same because of (b); by definition, y can change only when new values of x_t are observed, so that y remains the same. If (a) or (b) fails, D might well wish to resume sampling, as would be true, for example, if there was a cyclical time pattern in the distribution of x_t—say, the price distribution for a seasonal good—or if there was a time-dependent decrease in search costs.

[4] The only qualifications are these: (a) As noted in the text, Theorem 5 and Corollary 6 do not hold if there is no recall. (b) Remark 7 is obviously false if the sampling horizon is finite and similarly for Remark 8 if the sampling horizon is finite and there is no recall. (c) The formulas in Lemma 9, Remark 11, and Remark 12 must be modified slightly if the horizon is finite. (d) The sampling horizon must be two periods or more for Remark 18 to apply.

In order to determine the value of a particular policy d, define the sets $d_i = \{x \mid d(x) = i\}$. That is, d_i is the set of all infinite sequences for which D would stop immediately after i draws. If $d(x) = i$, then D stops on the ith successive draw after the decision point under consideration. He then goes on to enjoy, for every period after the ith, the best of $u(x_1),..., u(x_i)$ and y, the best utility available *at* the decision point. Therefore, if $d(x) = i$, D enjoys

$$\frac{k_1(x_1)}{(1+r)^1} + \cdots + \frac{k_i(x_i)}{(1+r)^i} + \max(y, u(x_1),..., u(x_i)) \left[\frac{1}{(1+r)^{i+1}} + \frac{1}{(1+r)^{i+2}} + \cdots\right]$$

$$= \frac{k_1(x_1)}{(1+r)^1} + \cdots + \frac{k_i(x_i)}{(1+r)^i} + \frac{\max(y, u(x_1),..., u(x_i))}{(1+r)^i\, r}. \tag{2}$$

The expected value $W(y, F, d)$ of a policy $d \not\equiv 0$, given y and F, is obtained by integrating the above expression (for all i) over all possible infinite sequences x with respect to the probability measure induced by F:

$$W(y, F, d) = \sum_{i=1}^{\infty} \int_{d_i} \frac{k_1(x_1)}{(1+r)^1} + \cdots + \frac{k_i(x_i)}{(1+r)^i} + \frac{\max(y, u(x_1),..., u(x_i))}{(1+r)^i\, r}\, dF(x). \tag{3}$$

The value of $d \equiv 0$, stopping immediately and taking no further samples, is just

$$W(y, F, d \equiv 0) = \frac{y}{(1+r)^1} + \frac{y}{(1+r)^2} + \cdots = \frac{y}{r}. \tag{4}$$

We define $V(y, F) = \max_d W(y, F, d)$ as the value of the optimal policy. Of course, neither the existence of an optimal policy nor its finite value can be assumed. In Appendix A we prove that an optimal and finite-valued policy does exist under quite general conditions, for instance when utility is bounded. From the above definitions we have

Remark 0. *$V(y, F)$ is nondecreasing in y since $W(y, F, d)$ is obviously nondecreasing in y for any d.*

The remainder of this section is devoted to characterizing the optimal policy. At a decision point, D's immediate problem is whether to continue sampling. In Theorem 4 we show that the optimal decision rule for this problem is to continue sampling if D's current best available utility is less than a switchpoint level of utility s. This rule completely characterizes

D's optimal policy because *any decision to be made in the future, say whether to sample the $(n+1)$-st time, may be viewed as a "current" decision, current when D has just made the nth draw.*

Theorem 4 is proved in several steps. First, we show that if it is optimal to stop with a given best available level of utility, it is certainly optimal to stop with anything better (Lemma 1). We then show that there can exist at most one best available level of utility at which D is indifferent between sampling and stopping (Corollary 2). Next, we show that there exist (except in a special case) levels of available utility so low as to induce sampling and other levels so high as to make stopping the optimal policy (Lemma 3). Finally, we use a continuity argument to prove that there exists a unique switchpoint or indifference level of best available utility s such that, at levels below s, D prefers to sample and, at levels above s, he prefers to stop (Theorem 4).

LEMMA 1. *Given the probability distribution F, if D is indifferent between stopping and sampling again or if he strictly prefers to stop when the best available utility is y, then D will strictly prefer to stop when the best available utility is y', for y' greater than y.*

Proof. Assume otherwise, that at y' there is a policy d' which involves sampling at least once and which is at least as good as stopping immediately. That is,

$$W(y', F, d') \geqslant W(y', F, d = 0) = \frac{y'}{r}. \tag{5}$$

Using (3), we have

$$\begin{aligned}
&W(y', F, d') - W(y, F, d') \\
&= \sum_{i=1}^{\infty} \int_{d_i'} \frac{\max(y', u(x_1),..., u(x_i)) - \max(y, u(x_1),..., u(x_i))}{(1+r)^i\, r}\, dF(x) \\
&\leqslant \sum_{i=1}^{\infty} \int_{d_i'} \frac{y' - y}{(1+r)^i\, r}\, dF(x) \\
&< \sum_{i=1}^{\infty} \int_{d_i'} \frac{y' - y}{r}\, dF(x) = \frac{y' - y}{r} \sum_{i=1}^{\infty} \int_{d_i'} dF(x) \\
&= \frac{y' - y}{r}.
\end{aligned} \tag{6}$$

Therefore

$$W(y, F, d') > W(y', F, d') - \frac{y' - y}{r}$$
$$\geqslant \frac{y}{r} \tag{7}$$

(the last inequality follows from (5)). This contradicts the optimality of stopping at y, the value of which is y/r. Q.E.D.

COROLLARY 2. *If there exists a level of best available utility y^* at which D is indifferent between stopping and sampling again, then D must strictly prefer to sample when the best available utility is less than y^* and he must strictly prefer to stop when the best available utility is greater than y^*. It follows that y^* is unique.*

Proof. Suppose that the best available utility is y and that y is greater than y^*. Then by Lemma 1, D strictly prefers to stop. Suppose that y is less than y^*. If D is indifferent or prefers to stop, then by Lemma 1, he strictly prefers to stop at y^*. This is a contradiction. Hence, D must strictly prefer to sample for y less than y^*. Q.E.D.

LEMMA 3. *There always exists a level of best available utility y^+ so high that D will prefer to stop. For the experience case there also exists a level of best available utility y^- so low that D will strictly prefer to sample. However, for the inspection case such a y^- may not exist if utility is bounded below and if the sampling cost is sufficiently great.*

Proof. (i) y^+: Suppose that utility is unbounded and consider the function $W(y, F, d')$ for given F and given $d' \not\equiv 0$. Then it can be shown (see Appendix B) that for any y'

$$W(y, F, d') \leqslant W(y', F, d') + \frac{y - y'}{r(1 + r)} \quad \text{for} \quad y \geqslant y'. \tag{8}$$

This is illustrated in Fig. 1. If it is optimal to stop at y' ($V(y', F) = y'/r$), then by Lemma 2 any $y^+ > y'$ will serve. If it is optimal to continue, then the line $V(y', F) + (y - y')/(r(1 + r))$ lies above $W(y, F, d)$ for any $y > y'$ and for *all* $d \not\equiv 0$; this follows from $V(y', F) \geqslant W(y', F, d)$ and from (8). In this case any $y^+ > P$ in Fig. 1 will do.

For bounded utility let $L = \sup_x u(x)$. If L is achievable, let $y^+ = L$. If not, it is possible to choose an ϵ such that $y^+ = L - \epsilon$.[5]

[5] Choose ϵ so that $(L - \epsilon)/r > E_F 1(u(x_1))/(1 + r) + L/(r(1 + r))$. Such an ϵ exists as $E_F 1(u(x_1)) < L$. The right-hand side is, in turn, greater than the value of sampling once and then continuing optimally.

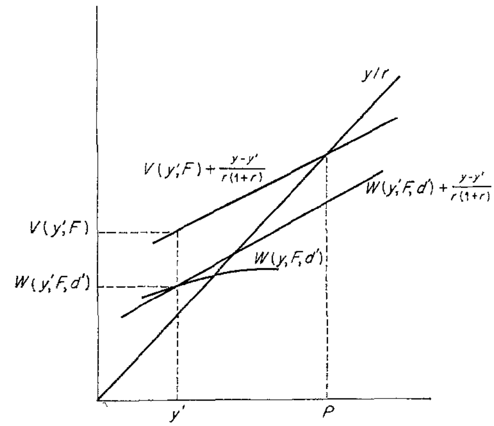

FIGURE 1

(ii) y^- for the experience case: Suppose that y is the best available utility and that $y < E[u(x_1)]$, where the expectation is over the marginal distribution for x_1, the first sample. Consider the policy $d \equiv 1$, sample once only.

$$W(y, F, d \equiv 1) = E\left[\frac{u(x_1)}{1+r} + \frac{\max(y, u(x_1))}{(1+r)r}\right]$$

$$\geqslant E\left[\frac{u(x_1)}{1+r} + \frac{u(x_1)}{(1+r)r}\right] = \frac{E[u(x_1)]}{r}$$

$$> \frac{y}{r} = W(y, F, d \equiv) 0. \tag{9}$$

Hence stopping immediately is not optimal and D will sample at least once for such a y.

(iii) y^- for the inspection case: The value of sampling once and then doing what is optimal is (by the logic of (B.1) in Appendix B)

$$\frac{-c + E_F{}^1 V(\max(y, u(x_1)), F^{x_1})}{1+r}, \tag{10}$$

where $c = Ec(1)$ and where F^{x_1} is the *joint* distribution (over all sequences of draws $(x_2, x_3, \ldots)$) as revised in view of the outcome of the first draw; the value of stopping immediately is y/r.

Suppose utility is bounded below by b. Then, since the expectation in (10) is finite for every y (if it were infinite $V(y, F)$ would be infinite, but $V(y, F)$ is finite), it is finite for $y = b$. Therefore, we can choose c

so large that (10) is less than b/r, meaning that D will stop if $y = b$. By Lemma 1, D will stop for $y \geqslant b$ and hence there is no attainable y^- in this case.

Suppose utility is not bounded below. We wish to show that for any finite c, a y^- exists. Let

$$\frac{y^-}{r} < \frac{Eu(x_1)}{r(1+r)} - \frac{c}{1+r},$$

where the expectation is over the marginal distribution for x_1. Consider the policy $d \equiv 1$.

$$W(y, F, d \equiv 1) = E\left[\frac{-c}{1+r} + \frac{\max(y, u(x_1))}{(1+r)r}\right]$$

$$\geqslant E\left[\frac{-c}{1+r} + \frac{u(x_1)}{(1+r)r}\right] > \frac{y^-}{r}. \qquad (11)$$

Hence it must be optimal to sample at least once at y^-. Q.E.D.

THEOREM 4. *Let D have a probability distribution F over future observations of x_t and let y be the best available utility. Then there exists a switch-point level of utility $s(F)$ such that D will strictly prefer to sample for $y < s(F)$, will be indifferent between sampling and stopping for $y = s(F)$, and will strictly prefer to stop for $y > s(F)$. For the inspection case, $s(F)$ may equal $-\infty$ if sampling costs are high enough and if utility is bounded below.*

Proof. Let $W(y, F) = \max_{d \neq 0} W(y, F, d)$ (see Appendix A for a discussion of the existence of $W(y, F)$). Note that $W(y, F)$ is continuous in y, being the maximum of a family of equicontinuous functions.[6] Therefore $g(y) = y/r - W(y, F)$ is continuous. By Lemma 3, there exist values y^+ and y^- (y^- may be $-\infty$ in the inspection case) such that $g(y^+) > 0$ and $g(y^-) < 0$. Hence g has at least one root. By definition of g, such a root is a point of indifference between stopping and sampling. By Corollary 2, there is at most one such point. Thus, there is exactly one root $s(F)$ where $g(s(F)) = 0$ and D is indifferent between stopping and sampling. Q.E.D.

Theorem 4 does *not* imply that D sets a critical level L for his *next* draw x_1 such that if $\max(y, u(x_1)) > L$ he stops and otherwise he continues. Indeed, as an example will show, the optimal policy of a rational individual does not always take the following form.

[6] By Remark 0 and (7), $|W(y, F, d) - W(y', F, d)| \leqslant |y - y'|/r$ for any y, y', F, and d. Thus, the family $W(\cdot, F, d)$ is equicontinuous, and it can easily be shown that the maximum of such a family of functions is itself a continuous function.

D knows that he is sampling from one of two iid distributions:

(A) $\text{Prob}(x_1 = 1) = 1$

and

(B) $\text{Prob}(x_1 = 2) = 0.0001,$
$\text{Prob}(x_1 = 3) = 0.9999.$

If $x_1 = 1$, he will wish to stop, if $x_1 = 2$, he will wish to continue (assuming low search costs, low rate of time preference and linear utility) and, if $x_1 = 3$, he will once again wish to stop.

Theorem 4 states that *after* D has drawn an observation and revised his subjective probability distribution, he sets a critical level such that if the utility of the best past observation (including the most recent) is greater than the critical level, he will stop sampling.

While in practice it is hard or impossible to solve explicitly for the function $s(F)$ described in Theorem 4, we do derive calculable upper and lower bounds for it in the following section.[7] However, if $s(F)$ were known, and given y and F, the following procedure could be used to construct $d^*(x)$, where d^* is the optimal policy and $x = (x_1, x_2, x_3, ...)$ is an arbitrary sequence of future draws. First, construct the sequence of currently best available utilities $Y = (y_0, y_1, y_2, ...)$ using the recursive formula $y_n = \max(y_{n-1}, u(x_n))$ and $y_0 = y$. That is, y_n is the best available utility after the nth draw. Similarly, construct the sequence of posterior distributions $\mathscr{F} = (F_0, F_1, F_2, ...)$, where F_n is the distribution F_{n-1} revised after observing x_n and where F_0 is F. For example, $F_1 = F^{x_1}$. Finally, construct the sequence of switchpoints $\mathscr{S} = (s_0, s_1, s_2, ...)$, where $s_n = s(F_n)$. Then in order to determine when to stop (i.e., to find $d^*(x)$), we compare the sequences Y and $\mathscr{S}$. Specifically, $d^*(x) = i$ if and only if $y_j < s_j$ for $j < i$ and $y_i \geqslant s_i$.

3. The Switchpoint

In this section we will find calculable bounds for the adaptive switchpoint.[8] The first step will be to relate the adaptive and static switchpoints.

[7] However, $s(F)$ can be approximated by solving the functional equation (12) by an iterative numerical procedure. The advantage of the upper and lower bounds which we derive is that they can be understood on intuitive grounds and that in calculating them one needs only to find the roots of monotonic functions.

[8] Our upper and lower bounds are similar in spirit to Yahav's [11] "optimist" and "conservative" stopping rules.

To this end, consider the functional equation

$$V(y, F) = \max\left(\frac{1}{1+r} E\{k_1(x_1) + V(\max(y, u(x_1)), F^{x_1})\}, \frac{y}{r}\right). \quad (12)$$

The first argument of the maximum function is, by the logic of (B.1) in Appendix B, the expected value of sampling once and then doing what is optimal. The second argument is the value of stopping immediately.

In order to evaluate $V(\max(y, u(x_1)), F^{x_1})$, it is useful to partition the set of possible realizations of x_1 into four subsets. Having observed x_1, it will either be optimal to stop or it will not; in either case $u(x_1)$ is greater than or equal to y or it is not. This two-way categorization induces the partition on the possible x_1 so that

$$V(\max(y, u(x_1)), F^{x_1}) = \begin{cases} y/r & \text{if } x_1 \in \mathrm{SL}(y), \\ u(x_1)/r & \text{if } x_1 \in \mathrm{SG}(y), \\ V(y, F^{x_1}) & \text{if } x_1 \in \mathrm{CL}(y), \\ V(u(x_1), F^{x_1}) & \text{if } x_1 \in \mathrm{CG}(y), \end{cases} \quad (13)$$

where

$$\mathrm{SL}(y) = \{x_1 \mid s(F^{x_1}) \leqslant \max(y, u(x_1)); y \geqslant u(x_1)\}$$

(it is optimal to stop; $u(x_1)$ is less than or equal to y);

$$\mathrm{SG}(y) = \{x_1 \mid s(F^{x_1}) \leqslant \max(y, u(x_1)); y < u(x_1)\}$$

(it is optimal to stop; $u(x_1)$ is greater than y);

$$\mathrm{CL}(y) = \{x_1 \mid s(F^{x_1}) > \max(y, u(x_1)); y \geqslant u(x_1)\}$$

(it is optimal to continue; $u(x_1)$ is less than or equal to y);

$$\mathrm{CG}(y) = \{x_1 \mid s(F^{x_1}) > \max(y, u(x_1)); y < u(x_1)\}$$

(it is optimal to continue; $u(x_1)$ is greater than y); and

$$\mathrm{G}(y) = \mathrm{SG}(y) \cup \mathrm{CG}(y) = \{x_1 \mid y < u(x_1)\}.$$

Since D is indifferent between sampling again and stopping at the switchpoint, we have from (12)

$$V(s(F), F) = \frac{1}{1+r} E\{k_1(x_1) + V(\max(s(F), u(x_1)), F^{x_1})\} = \frac{s(F)}{r}. \quad (14)$$

Rearranging this, we obtain

$$s(F) - Ek_1(x_1) = E\left\{V(\max(s(F), u(x_1)), F^{x_1}) - \frac{s(F)}{r}\right\}, \quad (15)$$

which, by (13), becomes

$$\begin{aligned}
s(F) - Ek_1(x_1) = &\int_{\mathrm{SL}(s(F))} \left[\frac{s(F)}{r} - \frac{s(F)}{r}\right] dF^1(x_1) \\
&+ \int_{\mathrm{SG}(s(F))} \left[\frac{u(x_1)}{r} - \frac{s(F)}{r}\right] dF^1(x_1) \\
&+ \int_{\mathrm{CL}(s(F))} \left[V(s(F), F^{x_1}) - \frac{s(F)}{r}\right] dF^1(x_1) \\
&+ \int_{\mathrm{CG}(s(F))} \left[V(u(x_1), F^{x_1}) - \frac{s(F)}{r}\right] dF^1(x_1) \\
= &\int_{\mathrm{SG}(s(F))} \left[\frac{u(x_1) - s(F)}{r}\right] dF^1(x_1) \\
&+ \int_{\mathrm{CL}(s(F))} \left[V(s(F), F^{x_1}) - \frac{s(F)}{r}\right] dF^1(x_1) \\
&+ \int_{\mathrm{CG}(s(F))} \left[V(u(x_1), F^{x_1}) - \frac{s(F)}{r}\right] dF^1(x_1). \qquad (16)
\end{aligned}$$

For the static case (independent and identically distributed draws from a known distribution), $F^{x_1} = F$ by definition. It should be clear that in this case the sets $\mathrm{CG}(s(F))$ and $\mathrm{CL}(s(F))$ are null: $[x_1 \in \mathrm{CG}(s(F))$ or $x_1 \in \mathrm{CL}(s(F))] \Leftrightarrow s(F^{x_1}) = s(F) > \max(s(F), u(x_1))$. Also, for the static case, $\mathrm{SG}(s(F)) = G(s(F))$, so that (16) reduces to

$$s(F) - Ek_1(x_1) = \int_{\mathrm{G}(s(F))} \left[\frac{u(x_1) - s(F)}{r}\right] dF^1(x_1). \qquad (17)$$

Using (16) and (17)—we postpone their interpretation for the moment—it is possible to determine a lower bound for the adaptive switchpoint.

THEOREM 5. *The adaptive switchpoint $s(F)$ is greater than or equal to the static switchpoint associated with F^1 (the marginal distribution from F over the next draw).*

Proof. Let $s_A = s(F)$, the adaptive switchpoint, and let s_N be the static switchpoint when sampling an iid sequence of random variables with common distribution F^1.

From (16) we have

$$\begin{aligned}
s_A - Ek_1(x_1) = &\int_{\mathrm{SG}(s_A)} \left[\frac{u(x_1) - s_A}{r}\right] dF^1(x_1) \\
&+ \int_{\mathrm{CL}(s_A)} \left[V(s_A, F^{x_1}) - \frac{s_A}{r}\right] dF^1(x_1) \\
&+ \int_{\mathrm{CG}(s_A)} \left[V(u(x_1), F^{x_1}) - \frac{s_A}{r}\right] dF^1(x_1). \qquad (18)
\end{aligned}$$

Since $V(s_A, F^{x_1}) \geqslant s_A/r$, the second integral above is nonnegative, so that

$$s_A - Ek_1(x_1) \geqslant \int_{SG(s_A)} \left[\frac{u(x_1) - s_A}{r}\right] dF^1(x_1) + \int_{CG(s_A)} \left[V(u(x_1), F^{x_1}) - \frac{s_A}{r}\right] dF^1(x_1). \tag{19}$$

Also, since $V(u(x_1), F^{x_1}) \geqslant u(x_1)/r$, this becomes

$$s_A - Ek_1(x_1) \geqslant \int_{SG(s_A) \cup CG(s_A)} \left[\frac{u(x_1) - s_A}{r}\right] dF^1(x_1) = \int_{G(s_A)} \left[\frac{u(x_1) - s_A}{r}\right] dF^1(x_1). \tag{20}$$

Now assume that $s_A < s_N$ and therefore $G(s_N) \subset G(s_A)$. Then

$$s_A - Ek_1(x_1) \geqslant \int_{G(s_A)} \left[\frac{u(x_1) - s_A}{r}\right] dF^1(x_1) \geqslant \int_{G(s_N)} \left[\frac{u(x_1) - s_N}{r}\right] dF^1(x_1) = s_N - Ek_1(x_1), \tag{21}$$

since s_N is the static switchpoint. But (21) implies that $s_A \geqslant s_N$, which is contrary to our assumption. Therefore, s_A must be at least as great as s_N.
Q.E.D.

This lower bound is indeed calculable as the static switchpoint is the root of a known monotonic function (from (17)).

Theorem 5 may be reinterpreted as a comparison of the behavior of two individuals, A and N, sharing the same utility function and rate of time preference but differing in their beliefs about the sampling distribution. A is adaptive and has a subjective probability distribution F over future draws; N believes the marginal distribution for the next draw to be F^1, as does A, but believes that all future draws are iid with this same distribution (the static case). In this context the theorem may be restated as saying that *A's switchpoint, s_A, is at least as great as that of N, s_N.*

This result can be understood in terms of the relative costs and benefits of sampling and stopping. At the switchpoints the expected immediate cost of continued sampling equals the expected future gain for both A and N. The expected immediate cost of one more draw consists of the foregone enjoyment of $s(F)$ for one period plus the actual cost of searching, $-Ek_1(x_1)$. For N the expected future gain consists of the possibly enhanced flow of utility in periods after the next, resulting from the possible observation of a value of x_1 better than $s(F)$. A has another source of potential gain: information leading to a revised distribution, F^{x_1}, that is more favorable than F in the sense that it warrants further sampling (this occurs if $x_1 \in \mathrm{CG}(S(F))$ or $x_1 \in \mathrm{CL}(s(F))$). *Therefore, subjective uncertainty about the "true" distribution may be preferable, in a limited sense, to certain knowledge*; for A the prospect of good news about the distribution outweighs the prospect of bad, as he is "insured" against the latter by the privilege of recall (without the privilege of recall the last theorem and the following corollary do not hold). Hence,

COROLLARY 6. *Suppose that s_A is greater than s_N (only in pathological cases does s_A equal s_N). Then certainly for $y \in [s_N, s_A)$ the value to A of A's optimal policy, $V_A(y)$, must be greater than the value to N of N's optimal policy, $V_N(y)$.*

Proof. Since A strictly prefers to continue and N prefers to stop,

$$V_A(y) > y/r = V_N(y) \qquad \text{for} \quad y \in [s_N, s_A). \tag{22}$$

Q.E.D.

Before deriving an upper bound for the adaptive switchpoint, we state two simple results about static behavior which follow immediately from the characterization of the static switchpoint given by (17).

Remark 7. If search costs do not depend on the index of the draw j, the privilege of recall will never be used in the static case.[9]

Recall clearly does not matter in this case as s is constant and search ends with the first observed value greater than s. If s ever falls, as may occur in the static case when search costs rise or in the adaptive case when unfavorable evidence accumulates, recall may be used.

Remark 8. Let $s = s(F)$ be the switchpoint associated with a static distribution F. Then arbitrary changes in this distribution over the set of x_1 no better than s which leave $Ek_1(x_1)$ fixed do not alter the switchpoint. In the inspection case, $Ek_1(x_1)$ is independent of x_1, so that *any* change in F over the set of x_1 no better than s leaves the switchpoint unchanged.

This follows from (17), which depends on the above set only through $Ek_1(x_1)$.

An example may help to illustrate the meaning of this remark. Suppose D has a used car to sell, pays a fixed cost to solicit a bid, *knows* the distribution of possible bids, and has calculated a switchpoint price of,

[9] This result is well known and is mentioned, for instance, in DeGroot [5].

say, \$500. Then the remark implies that arbitrary changes in the distribution of bids below \$500 do not matter, in that D's switchpoint does not change. Thus, if the probability of a bid below \$500 remains the same, say at 0.4, it makes no difference whether this probability is distributed uniformly (or otherwise) over the interval \$0–500 or is concentrated at \$1 or even at \$499.

We derive a calculable upper bound for the adaptive switchpoint in four steps. (1) We find a formula for the value of the optimal policy in the static case (Lemma 9). (2) Using this, we show that the value of continuing to search in the adaptive case, given a best observed value y, is less than $Z(y)$, the value of the hypothetical sequence of sampling once, of being told the true distribution, and then of proceeding optimally (Remark 10). (3) We prove that if the value of stopping, y/r, is greater than $Z(y)$, then it is optimal to stop (Remark 11). (4) We show that s^* such that $s^*/r = Z(s^*)$ is an upper bound for the adaptive switchpoint (Corollary 12).

LEMMA 9. *For the static case, if search costs do not depend on the index of the draw, j, then*

$$V(y, F) = \begin{cases} s(F)/r & y < s(F), \\ y/r & y \geqslant s(F). \end{cases} \tag{23}$$

Proof. Let $P(s) = \text{Prob}\{x_1 \mid u(x_1) \leqslant s(F)\}$. The expected value of the search using the optimal switchpoint $s = s(F)$ equals the expected payoff if D stops after one trial plus the expected payoff, if D stops after two trials plus ... . Therefore, for $y < s$,

$$\begin{aligned} V(y, F) &= \frac{1}{1+r}\left[Ek_1(x_1) + \int_{G(s)} \frac{u(x_1)}{r}\, dF^1(x_1)\right] \\ &\quad + \frac{P(s)}{(1+r)^2}\left[Ek_1(x_2) + \int_{G(s)} \frac{u(x_2)}{r}\, dF^1(x_2)\right] \\ &\quad + \frac{P(s)^2}{(1+r)^3}\left[Ek_1(x_3) + \int_{G(s)} \frac{u(x_3)}{r}\, dF^1(x_3)\right] + \cdots \\ &= \frac{Ek_1(x_1) + \int_{G(s)} [u(x_1)/r]\, dF(x_1)}{1 + r - P(s)} = \frac{s}{r} \end{aligned} \tag{24}$$

The last step makes use of (17). Q.E.D.

To illustrate the lemma, suppose that D is selling his car, knows the distribution of possible offers, and determines that the utility of \$500 is his switchpoint. Then he would be just as well off having received a maximum bid of \$1 as he would having received a maximum bid of \$499.

For the rest of this section we restrict our attention to Bayesian learning about an unknown parameter vector θ of the common distribution of a sequence of independent and identically distributed random variables. $G(\theta)$ is the prior distribution over the unknown parameter vector.

Remark 10. When the best observed value is y, $Z(y)$ is at least as great as the value of continuing to sample.

Proof. $Z(y)$, the expected value of the hypothetical sequence of sampling once, of being told the true distribution, and then of proceeding optimally, is given by

$$Z(y) = \frac{1}{1+r}\Big[Ek_1(x_1) + \iint \max_d W(\max(y, u(x_1)), F_\theta, d)\, dG(\theta \mid x_1)\, dF^1(x_1)\Big], \quad (25)$$

where F_θ is the distribution over infinite sequences $(x_2, \ldots)$ when the parameter is θ; where $\max_d W(\max(y, u(x_1)), F_\theta, d)$ is the value of the best policy when x_1 has been drawn and the parameter is known to be θ; and where $G(\theta \mid x_1)$ is the conditional prior over θ given x_1. But

$$\begin{aligned} &\iint \max_d W(\max(y, u(x_1)), F_\theta, d)\, dG(\theta \mid x_1)\, dF^1(x_1) \\ &\quad \geqslant \int \max_d \int W(\max(y, u(x_1)), F_\theta, d)\, dG(\theta \mid x_1)\, dF^1(x_1) \\ &\quad = E_{F^1} V(\max(y, u(x_1)), F^{x_1}), \end{aligned} \quad (26)$$

which is the undiscounted expected value of continuing optimally after sampling once. Hence,

$$Z(y) \geqslant \frac{1}{1+r}\,[Ek_1(x_1) + E_{F^1} V(\max(y, u(x_1)), F^{x_1})], \quad (27)$$

the value of sampling once and then proceeding optimally. Q.E.D.

Since $\max_d W(\max(y, u(x_1)), F_\theta, d)$ is the value of following the optimal policy for a static distribution, we may use Lemma 9 to rewrite (25) as

$$\begin{aligned} Z(y) &= \frac{1}{1+r}\Big[Ek_1(x_1) + \frac{1}{r}\iint \max(s_\theta, y, u(x_1))\, dG(\theta \mid x_1)\, dF^1(x_1)\Big] \\ &= \frac{1}{1+r}\Big[Ek_1(x_1) + \frac{1}{r}\iint \max(s_\theta, y, u(x_1))\, dH(x_1, s_\theta)\Big]. \end{aligned} \quad (28)$$

$H(\cdot, \cdot)$ is the joint distribution of x_1 and of s_θ, the static switchpoint associated with the distribution F_θ (i.e., s_θ satisfies

$$s_\theta - Ek_1(x_1) = \frac{1}{r} \int_{u(x_1) > s_\theta} [u(x_1) - s_\theta] \, dF_\theta{}^1(x_1) \tag{29}$$

as in (17) above).

Equation (28) involves only static switchpoints, so that $Z(y)$ may be calculated.

Remark 11. If $y/r \geqslant Z(y)$, then it is optimal to stop.

Proof. y/r is the value of stopping. If $y/r \geqslant Z(y)$, then, by Remark 10, y/r exceeds the value of continuing to sample. Hence it is optimal to stop. Q.E.D.

The procedure of stopping when $y/r \geqslant Z(y)$ will never cause D to stop prematurely but it will allow him to continue sampling too long.

Remark 12. s^* satisfying

$$s^*/r = Z(s^*) \tag{30}$$

is an upper bound for the true adaptive switchpoint s.

Proof. By Remark 11, for any y such that $y/r \geqslant Z(y)$, it is optimal to stop. According to Theorem 4, any y for which it is optimal to stop must be greater than or equal to s. Since s^* satisfies $s^*/r \geqslant Z(s^*)$, it too is greater than or equal to s and so is an upper bound for s. Q.E.D.

The upper bound s^* may itself be interpreted as a switchpoint. From (28) and (30) we have

$$\frac{s^*}{r} = \frac{1}{1+r} \left[Ek_1(x_1) + \frac{1}{r} \iint \max(s_\theta, s^*, u(x_1)) \, dH(x_1, s_\theta) \right] \tag{31}$$

or

$$s^* - Ek_1(x_1) = \frac{1}{r} \iint_{\max(u(x_1), s_\theta) > s^*} [\max(u(x_1), s_\theta) - s^*] \, dH(x_1, s_\theta). \tag{32}$$

To reiterate, from F, D's current distribution over future sequences, it is possible to calculate two numbers, s_N and s^*, such that if the best observed y is less than s_N, D should continue to sample, and if y is greater than s^*, D should stop.[10] Note that in the static case, when there is nothing to be learned, the upper and lower bounds coincide with the switchpoint.

4. Time Preference, Search Costs, Risk Aversion, and the Probability Distribution

For every result on changes in the switchpoint (such as those of this section) there is, as a trivial corollary, a corresponding result on changes in the duraction of search: If individuals A and B begin with the same prior F and draw identical samples $x_1, x_2, \ldots$, yet after each draw B has a higher switchpoint than A (perhaps because A is more impatient, pays a higher search cost, or is more risk averse), then by definition of the switchpoint A cannot continue to sample after B has stopped. *Thus, higher switchpoints mean more search and, in particular, a higher expected duration of search.* One can calculate by how much if one can find an explicit relationship between the duration of search and the switchpoint. This seems possible only in the static case with constant search costs, when the expected duration of search is $1/(1 - P(s))$.

Theorem 13. *The switchpoint s falls with a rise in the rate of time preference r.*

Proof. Let the switchpoint $s(F) = s$ so that

$$\frac{s}{r} = V(s, F; r) = W(s, F, d^*; r), \tag{33}$$

where D is indifferent between stopping and the rule $d^* \neq 0$. From (3) and (33) we have

$$\begin{aligned} 0 &= W(s, F, d^*; r) - \frac{s}{r} \\ &= \sum_{i=1}^{\infty} \int_{d_i^*} \Big[\frac{k_1(x_1) - s}{(1 + r)} + \cdots + \frac{k_i(x_i) - s}{(1 + r)^i} \\ &\quad + \frac{\max(s, u(x_1), \ldots, u(x_i)) - s}{(1 + r)^i\, r}\Big]\, dF(x) \end{aligned} \tag{34}$$

[10] In a forthcoming paper we will discuss properties of these bounds and report some experiments with actual distributions.

and

$$\frac{d}{dr}\left[W(s, F, d^*; r) - \frac{s}{r}\right]$$

$$= -\frac{1}{1+r}\sum_{j=1}^{\infty}\sum_{i=j}^{\infty}\int_{d_i^*}\left[\frac{k_j(x_j) - s}{(1+r)^j} + \cdots + \frac{k_i(x_i) - s}{(1+r)^i}\right.$$

$$\left. + \frac{\max(s, u(x_1),..., u(x_i)) - s}{(1+r)^i\, r}\right] dF(x)$$

$$- \frac{1}{r}\sum_{i=1}^{\infty}\int_{d_i^*}\left[\frac{\max(s, u(x_1),..., u(x_i)) - s}{(1+r)^i\, r}\right] dF(x). \qquad (35)$$

Note that

$$\sum_{i=j}^{\infty}\int_{d_i^*}\left[\frac{k_j(x_j) - s}{(1+r)^j} + \cdots + \frac{k_i(x_i) - s}{(1+r)^i}\right.$$

$$\left. + \frac{\max(s, u(x_1),..., u(x_i)) - s}{(1+r)^i\, r}\right] dF(x) \geqslant 0 \qquad \text{for all } j. \qquad (36)$$

For suppose that (36) is negative for some j. Then from (34) it follows that

$$\sum_{i=1}^{j-1}\int_{d_i^*}\left[\frac{k_1(x_1) - s}{(1+r)} + \cdots + \frac{k_i(x_i) - s}{(1+r)^i}\right.$$

$$\left. + \frac{\max(s, u(x_1),..., u(x_i)) - s}{(1+r)^i\, r}\right] dF(x)$$

$$+ \sum_{i=j}\int_{d_i^*}\left[\frac{k_1(x_1) - s}{(1+r)} + \cdots + \frac{k_j(x_{j-1}) - s}{(1+r)^{j-1}}\right] dF(x) > 0. \quad (37)$$

But (37) may be recognized as

$$W(s, F, d^{*j}; r) - \frac{s}{r} > 0, \qquad (38)$$

where d^{*j} is the policy "follow d^* for the first $j - 1$ periods and then, if you haven't stopped, take one more draw and stop without awarding yourself any stream of best available utility thereafter." Since, by (38), d^{*j} is better than $d \equiv 0$, the optimality of $d \equiv 0$ is contradicted and (36) must hold.

The last term in (35) must be less than zero. If it were not, then D

would be certain that in following d^*, $u(x_i) \leqslant s$ with probability one. Thus d^* would be worse than $d \equiv 0$, which is a contradiction. Hence

$$\frac{d}{dr}\left[W(s, F, d^*; r) - \frac{s}{r}\right] < 0. \tag{39}$$

Thus, if $r' < r$, $W(s, F, d^*; r') > s/r$, so that $d \equiv 0$ cannot be optimal. Therefore the new switchpoint must be higher than s. Q.E.D.

THEOREM 14. *The switchpoint falls with an increase in next-period expected search costs (from* (14)).[11]

THEOREM 15. *As the rate of time preference increases, the switchpoint level of utility falls to the expected utility of the next draw in the experience case and to the cost of sampling in the inspection case.*

Proof. It suffices to show that the right-hand side of (16) approaches zero as r increases. Clearly, the first of the three integrals on the right-hand side tends to zero. To show that the other two integrals tend to zero, consider $y \geqslant s(F)$ so that $V(y, F^{x_1}; r) - s(F)/r \geqslant 0$. Pick $y' \geqslant s(F^{x_1})$ and $y' \geqslant y$. Then, by Theorem 13, for all $r' \geqslant r$ we have $V(y', F^{x_1}; r') = y'/r'$. Therefore, since by Remark 0 $V(y', F^{x_1}; r') \geqslant V(y, F^{x_1}; r')$, we have

$$\frac{y'}{r'} - \frac{s(F)}{r'} \geqslant V(y, F^{x_1}; r') - \frac{s(F)}{r'} \geqslant 0 \qquad \text{for} \quad r' \geqslant r. \tag{40}$$

Clearly $y'/r' - s(F)/r'$ tends to zero as r' tends to infinity so that the same is true for $V(y, F^{x_1}; r') - s(F)/r'$. Therefore the integrands of the two integrals in question tend to zero pointwise, and by the Lebesgue convergence theorem this implies that the integrals themselves tend to zero. The theorem applies since the integrals are bounded by an integrable function (for instance by Z of Appendix A) as r tends to infinity. Q.E.D.

In order to examine the relationship between risk aversion and search behavior, consider two decisionmakers A and B who have identical rates of time preference and the same distribution over future draws but have different utility functions. The utility function of B is a strictly concave positive transformation of that of A, so that B is more risk averse than A.[12] Then we have

THEOREM 16. *Given A with utility function u and switchpoint s_A and the more risk averse B with utility function $g(u)$ (where g is a positive*

[11] It can be shown that if expected search costs rise in *any* future period, the switchpoint cannot rise.

[12] See Pratt [10].

concave transformation) and switchpoint s_B, *then B will search less than A in the following sense:* $s_B < g(s_A)$. *Equivalently,* $\{x_t \mid u(x_t) > s_A\} \subset \{x_t \mid g(u(x_t)) > s_B\}$.

Proof. Let $g(s) = s_B$. It will be enough to show that $s < s_A$, for then $s_B = g(s) < g(s_A)$. Let $d^* \not\equiv 0$ be such that

$$\frac{g(s)}{r} = V(g(s) = s_B, F; g(u(\cdot))) = W(g(s), F, d^*; g(u(\cdot))),$$

so that

$$\begin{aligned} 0 &= \sum_{i=1}^{\infty} \int_{d_i^*} \Big[\frac{g(k_1(x_1)) - g(s)}{(1+r)} + \cdots + \frac{g(k_i(x_i)) - g(s)}{(1+r)^i} \\ &\quad + \frac{g(\max(s, u(x_1),..., u(x_i))) - g(s)}{(1+r)^i\, r}\Big]\, dF(x) \\ &< \sum_{i=1}^{\infty} \int_{d_i^*} \Big[\frac{g'(s)(k_1(x_1) - s)}{(1+r)} + \cdots + \frac{g'(s)(k_i(x_i) - s)}{(1+r)^i} \\ &\quad + \frac{g'(s)(\max(s, u(x_1),..., u(x_i)) - s)}{(1+r)^i\, r}\Big]\, dF(x) \\ &= g'(s)\left[W(s, F, d^*; u(\cdot)) - \frac{s}{r}\right] \\ &\leqslant g'(s)\left[V(s, F; u(\cdot)) - \frac{s}{r}\right]. \end{aligned} \tag{41}$$

As $g'(s) > 0$, the last term in brackets is greater than zero. Hence $V(s, F; u(\cdot)) > s/r$ and therefore $s < s_A$. Q.E.D.

One would like to describe the effect on D's behavior of a change in the probability distribution. Because of the difficulty of characterizing a change in the distribution in the most general case, we restrict attention to the effect on the static switchpoint of a translation and of a change in risk of the single-period distribution. The distribution is also assumed to be univariate. Note that changes in the distribution may often be decomposed into a translation followed by a change in risk.

Suppose that D's utility is increasing in x_1. Then we have the following.

THEOREM 17. *A translaion to the right of the static distribution raises the switchpoint level of utility.*

Proof. It suffices to show that a translation to the right by a constant $h > 0$ will reduce the left-hand side of (17) and raise the right-hand side.

As $Ek_1(x_1)$ will rise, the left-hand side will fall. Let F_h^1 denote the translated single-period distribution function and suppose x_1' satisfies $u(x_1') = s(F)$. Then

$$\int_{x_1>x_1'} u(x_1)\, dF^1(x_1) = \int_{x_1>x_1'+h} u(x_1 - h)\, dF_h^1(x_1)$$

$$< \int_{x_1>x_1'+h} u(x_1)\, dF_h^1(x_1) \leqslant \int_{x_1>x_1'} u(x_1)\, dF_h^1(x_1), \quad (42)$$

so that the right-hand side rises. Q.E.D.

Two types of change in risk of a distribution have been defined in the literature. They are the Rothschild–Stiglitz mean-preserving increase in risk [12] and the more recent Diamond–Stiglitz mean-*utility*-preserving increase in risk [6]. A mean-preserving increase in risk may be desired by a risk averter even though it reduces the expected utility of any given draw. This is because it makes the probability of extremely favorable observations higher and, since the individual is able to sample *many* times, he may find it profitable to wait for such observations.

Remark 18. A mean-preserving increase in risk of the static distribution may either raise or lower the switchpoint level of utility.

This is demonstrated by the following two examples.

(a) Consider a mean-preserving spread of F^1 resulting in a new single-period distribution H^1 such that the distribution over $G(s(F^1))$ remains unchanged. If the sampling costs (the left-hand side of (17)) were not affected, then, by Remark 8, $s(H^1) = s(F^1)$. However, suppose that in the experience case $E_{H^1}u(x_1) < E_{F^1}u(x_1)$. Then we would have $s(H^1) < s(F^1)$ since the costs of sampling from H^1 would exceed the benefits at $s(F^1)$ and the switchpoint would have to fall to restore equality in (17).

(b) If, on the other hand, the mean-preserving spread led to an H^1 that gave greater total probability weight to $G(s(F^1))$ and if sampling costs were not affected (say in the inspection case), then at $s(F^1)$ the benefits would exceed the costs for the new distribution at $s(F^1)$ and s would have to rise to restore equality, giving $s(H^1) > s(F^1)$.

THEOREM 19. *A mean-utility-preserving increase in risk of the static distribution can only raise the switchpoint level of utility.*

Proof. In order to apply the Diamond–Stiglitz definition of an increase in risk, we will restrict our attention to a univariate distribution F^1 defined over the unit interval. Let z be a parameter of F^1 whose increase signifies an increase in riskiness. The expected value of a draw is independent of changes in z, by definition, so that the expected costs of sampling will not change even in the experience case. By differentiating the expected benefits of sampling (the right-hand side of (17)) with respect to z and then rewriting the integral using a change of variable, we have

$$\begin{aligned} &\frac{d}{dz}\left[\frac{1}{r}\int_{u(x)=s}^{x=1}(u(x)-s)\,dF^1(x,z)\right] \\ &=\frac{1}{r}\int_{u(x)=s}^{x=1}(u(x)-s)\,dF_z^1(x;z) \\ &=[(u(x)-s)\,F_z^1(x;z)]_{u(x)=s}^{x=1}-\int_{u(x)=s}^{x=1}F_z^1(x;z)\,d[u(x)-s] \\ &=0-\int_{u(x)=s}^{x=1}F_z^1(x;z)\,u'(x)\,dx\geqslant 0. \end{aligned} \tag{43}$$

The inequality follows from the conditions used to define a mean-utility-preserving increase in risk [6; p. 8, (12) and (13)]:

$$\begin{aligned} &\int_0^y F_z^1(x;z)\,u'(x)\,dx\geqslant 0 \qquad \text{for all } y, \\ &\int_0^1 F_z^1(x;z)\,u'(x)\,dx=0. \end{aligned} \tag{44}$$

Since an increase in risk raises the right-hand side of (17), there must be a compensating increase in s to restore equality. Q.E.D.

COROLLARY 20. *If the utility function is linear, a mean-preserving increase in risk can only raise the switchpoint level of utility.*

Neither definition of increasing risk in a single-period distribution carries over satisfactorily to a sequential problem if one believes that a risk-averter should not prefer an increase in risk. In a sense this criticism is unfair, and it might be argued that some more meaningful application of a mean-utility-preserving increase in riskiness is possible: for instance, one might apply it to the distribution of discounted payoffs for an optimal search rather than to the single-period distribution. Our only intention is to show that a simple-minded extension of the notion of increased riskiness to the sequential case is not possible as it is with risk aversion (for a discussion of the latter, see Neave [8]).

5. CONCLUSION

We hope to have succeeded in providing some of the basic results necessary for the use of search models in economic theory. We have shown that a well-behaved optimal decision rule does exist and that meaningful comparative static results are obtainable even when the decisionmaker learns from experience. The derivation of calculable bounds for the adaptive switchpoint may be of use in practical applications.

Appendix A

In this appendix we prove the existence of an optimal stopping rule in two classes of adaptive search problem in which rewards are discounted: (1) when there is Bayesian learning about an unknown parameter of the distribution of an iid sequence of random variables; (2) when utility is bounded.

Related results have been obtained by Blackwell [1], Chow and Robbins [2–4], and Yahav [13]. Blackwell considers Markov decision processes with a bounded reward function and discounting. He shows that an optimal policy exists when the number of possible actions is finite; this may be applied to adaptive search since only two actions are possible—stopping and continuing. Chow and Robbins prove the existence of an optimal stopping rule in a broad class of problems when rewards are positive, but not necessarily bounded, sampling cost is a positive constant, and there is no discounting. Yahav proves the existence of an optimal stopping rule in the inspection case of adaptive search when sampling cost is a positive constant and when there is no discounting. Both Yahav and Chow and Robbins consider only Bayesian learning about an unknown parameter of the distribution of an iid sequence of random variables.

Our second existence theorem is not restricted to Markov decision processes and so differs from Blackwell's result. Our first theorem differs from Chow and Robbins' and from Yahav's in that it deals with discounted rewards and does not insist on a constant cost of search.

The argument will be divided into two parts. The first is a modification of Chow and Robbins' work, as presented by De Groot [5, p. 345]; optimal rules are shown to exist, given two conditions which are rather difficult to interpret. The second part of the argument is a discussion of some circumstances under which these two conditions hold.

For any infinite sequence x define $P_n(x)$ as follows:

$$P_n(x) = \frac{k_1(x_1)}{1+r} + \cdots + \frac{k_n(x_n)}{(1+r)^n} + \frac{\max(y, u(x_1), \ldots, u(x_n))}{(1+r)^n\, r}.$$

Thus $P_n(x)$ is the discounted payoff received if D stops sampling after the nth observation of the infinite sequence. Let $P_0(x) \equiv y/r$ and let $P_\infty(x) = \sum_{i=1}^{\infty} k_i(x_i)/(1+r)^i$ be the discounted payoff, possibly infinite, that D receives if he continues to sample forever.

If D is blessed with perfect foresight (i.e., if he could peek at the *entire* sequence x before deciding whether and when to stop), then he could do no better than attain the quantity

$$Z(x) = \max(\sup_n |P_n(x)|, |P_\infty(x)|). \tag{A.1}$$

We assume the following conditions.

Condition 1.

$$E_F(Z) = M < \infty. \tag{A.2}$$

Condition 2.

$$\lim_n P_n(x) = P_\infty(x) \tag{A.3}$$

almost everywhere.

Let d be a measurable stopping rule (i.e., $\{x \mid d(x) = n\}$ is measurable for all n). Then the value of following d may be written

$$W(d) = \int P_{d(x)}(x)\, dF(x). \tag{A.4}$$

The existence of the integral follows from (A.2).

A policy d is defined to be *regular if*

$$d(x) > n \Rightarrow W(d \mid (x_1, ..., x_n)) > P_n(x). \tag{A.5}$$

Here $W(d \mid (x_1, ..., x_n))$ is the value of the policy d given that $(x_1, ..., x_n)$ has been observed. Formally,

$$W(d \mid (x_1, ..., x_n)) = \int_{\{x \mid d(x) > n\}} P_{d(x)}(x)\, dF(x \mid (x_1, ..., x_n)). \tag{A.6}$$

Thus a policy d is regular if whenever sampling is continued it is strictly beneficial to do so. It should be intuitively clear that, if a policy is not regular, then there exists an equally good regular policy.

LEMMA A.1. *If d is not regular, there exists a regular policy d' such that $W(d') \geqslant W(d)$.*

For a proof, see De Groot [5, pp. 288–289].

Given n policies $d^1, ..., d^n$, define a new policy called m_n under which another observation is taken at each stage if and only if another is to be taken under at least one of $d^1, \ldots, d^n$. That is,

$$m_n(x) = \max(d^1(x), \ldots, d^n(x)). \tag{A.7}$$

LEMMA A.2. *Let $d^1, \ldots, d^n$ be regular policies. Then m_n is regular and $W(m_n) \geqslant W(d^i)$ for $i = 1, \ldots, n$.*

For a proof, see De Groot [5, pp. 290–291].

Since we are trying to find a policy having *maximum* value, we may restrict our attention to regular policies (by Lemma A.1). Furthermore, we may disregard all policies worth less than P_0. Therefore, we define the set S such that

$$S = \{d \mid d \text{ is regular and } W(d) \geqslant P_0\}. \tag{A.8}$$

S is not empty since it includes $d \equiv 0$, and so

$$\sup_{d \in S} W(d) = \sup_{d} W(d) = V. \tag{A.9}$$

THEOREM A.3. *Given Conditions 1 and 2, there exists an optimal policy.*

Proof. We will show that there exists a policy d^* such that $W(d^*) = \sup_{d \in S} W(d) = V$.

Choose a sequence of policies $(d^1, d^2, \ldots)$ all in S such that $\lim_i W(d^i) = V$. If the policy m_i is as defined above, then, by Lemma A.2, $m_i \in S$ and $\lim_i W(m_i) = V$.

In addition, for any sequence x,

$$m_i(x) \leqslant m_{i+1}(x) \qquad \text{for all } i. \tag{A.10}$$

Let policy d^* be such that $d^*(x) = \sup_i m_i(x)$, and choose an arbitrary sequence x. If $d^*(x) = \sup_i m_i(x) = j < \infty$, then by (A.10) there is an N such that $m_i(x) = j$ for $i \geqslant N$. Thus $P_{d^*(x)}(x) = P_j(x) = \lim_i P_{m_i(x)}(x)$. If, on the other hand, $d^*(x) = \sup_i m_i(x) = \infty$, then by (A.3)

$$P_{d^*(x)}(x) = P_\infty(x) = \lim_i P_i(x) = \lim_i P_{m^i(x)}(x) \tag{A.11}$$

with probability one. Hence the sequence of functions $P_{m^i(\cdot)}(\cdot)$ converge pointwise to the function $P_{d^*(\cdot)}(\cdot)$ almost everywhere. Since $|P_{m_i(x)}(x)| \leqslant Z(x)$ and Z is integrable, the Lebesgue convergence theorem may be applied to give us our result:

$$W(d^*) = \int P_{d^*(x)}(x)\, dF(x) = \lim_i \int P_{m_i(x)}(x)\, dF(x) = \lim_i W(m_i) = V.$$

Q.E.D.

We now demonstrate that Conditions 1 and 2 are satisfied in several important cases.

THEOREM A.4. *If utility is bounded, then Conditions* 1 *and* 2 *are satisfied.*

Proof. (i) Let utility be bounded by B so that $|u(x)| \leqslant B$. Then $Z(x) \leqslant 2B/r$ and Z is integrable.

(ii) Let

$$S_n(x) = \sum_{i=1}^{n} \frac{k(x_i)}{(1+r)^i}. \tag{A.12}$$

Then

$$|P_n(x) - S_n(x)| = \left| \frac{\max(y, u(x_1),\ldots, u(x_n))}{r(1+r)^n} \right| \leqslant \frac{B}{r(1+r)^n}, \tag{A.13}$$

and so

$$\lim_n P_n(x) = \lim_n S_n(x) = P_\infty(x). \tag{A.14}$$

Q.E.D.

Suppose D is drawing independent and identically distributed random samples with common distribution $F(\cdot; w)$ and that he has a prior distribution over the parameter w. Let $m_2(w)$ be the second moment, given w, of the random function $u(x_i)$ and let $\mu_{|u|}(w)$ be the conditional mean of its absolute value.

THEOREM A.5. *If the expectation over* w *of* $m_2(w)$ *exists, then Conditions* 1 *and* 2 *are satisfied.*

Proof. (i) Clearly, if $c = \sup |c(j)|$,

$$Z(x) \leqslant \sum_{i=1}^{\infty} \frac{\max(|y|, |u(x_1)|,\ldots, |u(x_i)|) + |c|}{(1+r)^i} \leqslant \sum_{i=1}^{\infty} \frac{|y| + |u(x_1)| + \cdots + |u(x_i)| + |c|}{(1+r)^i} \tag{A.15}$$

Thus

$$E(Z \mid w) \leqslant \frac{|y|}{r} + \sum_{i=1}^{\infty} \left[\frac{i\mu_{|u|}(w)}{(1+r)^i}\right] + \frac{|c|}{r} < \infty. \tag{A.16}$$

Expectation may be taken within the summation sign since we are dealing with nonnegative functions. Also, note that with probability one $m_2(w) < \infty$ and

$$m_2(w) < \infty \Rightarrow \mu(w) < \infty \Leftrightarrow \mu_{|u|}(w) < \infty. \tag{A.17}$$

Because

$$E(Z) = E_w E(Z \mid w), \tag{A.18}$$

and since (A.16) is linear in $\mu_{|u|}(w)$, it is enough to establish that $E\mu_{|u|}(w) < \infty$. But this follows from our assumptions.

(ii) Fix w and suppress it in the notation. By the logic of part (ii) of Theorem A.4,

$$\frac{\max(y, u(x_1),\ldots, u(x_n))}{r(1+r)^n} \to 0 \Rightarrow P_n(x) \to P_\infty(x). \tag{A.19}$$

Hence, it suffices to prove that

$$\frac{\max(u(x_1),\ldots, u(x_n))}{(1+r)^n} \to 0. \tag{A.20}$$

We will use the Borel–Cantelli lemma to show that (A.20) is true with probability one.

For any $\epsilon > 0$, let $Q(\epsilon) = \{x \mid \exists N \ni u(x_t) \leqslant (1+\epsilon)^t \text{ for } t > N\}$. We now show that $\text{Prob}(Q(\epsilon)) = 1$.

Choose n_1 and n_2 so that

$$(1+\epsilon)^n > \mu, \qquad \forall n \geqslant n_1, \tag{A.21}$$

and

$$(1+\epsilon)^{n+1} - (1+\epsilon)^n > \sigma, \qquad n \geqslant n_2. \tag{A.22}$$

Let $m = \max(n_1, n_2)$; then from (A.21) and (A.22),

$$\begin{aligned}(1+\epsilon)^{m+j} - \mu = {} & [(1+\epsilon)^{m+j} - (1+\epsilon)^{m+j-1}] + \cdots \\ & + [(1+\epsilon)^{m+1} - (1+\epsilon)^m] + [(1+\epsilon)^m - \mu] > j\sigma.\end{aligned} \tag{A.23}$$

Therefore, for $j \geqslant 1$,

$$\begin{aligned}u(x_{m+j}) > (1+\epsilon)^{m+j} \Rightarrow u(x_{m+j}) - \mu > (1+\epsilon)^{m+j} - \mu > j\sigma \\ \Rightarrow |\, u(x_{m+j}) - \mu \,| > j\sigma.\end{aligned} \tag{A.24}$$

Hence, by (A.24) and by Chebychev's inequality

$$\Pr\{x \mid u(x_{m+j}) > (1+\epsilon)^{m+j}\} \leqslant \Pr\{x \mid \mid u(x_{m+j}) - \mu \mid > j\sigma\}$$
$$\leqslant \frac{1}{j^2}. \tag{A.25}$$

Define

$$A_n = \{x \mid u(x_n) > (1+\epsilon)^n\}. \tag{A.26}$$

Then, by (A.25)

$$\sum_{n=1}^{\infty} \Pr(A_n) = \sum_{n=1}^{m} \Pr(A_n) + \sum_{n=m+1}^{m} \Pr(A_n)$$
$$\leqslant m + \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty. \tag{A.27}$$

By the Borel–Cantelli lemma [11, p. 114], (A.27) implies

$$\Pr\left\{x \mid x \in \bigcap_{n=1}^{\infty} A_n\right\} = 0. \tag{A.28}$$

In addition, if $(n_1, n_2, \ldots)$ is an infinite subsequence of the positive integers, then

$$\sum_{i=1}^{\infty} \Pr(A_{n_i}) \leqslant \sum_{n=1}^{\infty} \Pr(A_n) < \infty, \tag{A.29}$$

so that, with probability one, any sequence x must belong to only a finite number of the A_n.

Now let $\epsilon = r/2$. Then, with probability one, there is an N such that $u(x_t) \leqslant (1 + r/2)^t$ for all $t > N$. Let $u^* = \max(u(x_1), \ldots, u(x_N))$. Then, for $t > N$,

$$\frac{\max(u(x_1), \ldots, u(x_t))}{(1+r)^t} \leqslant \frac{\max(u^*, (1+r/2)^{N+1}, \ldots, (1+r/2)^t)}{(1+r)^t}$$
$$\leqslant \frac{\max(u^*, (1+r/2)^t)}{(1+r)^t}, \tag{A.30}$$

which goes to zero as t increases.

Since we have shown that (A.20) holds for any w with probability one; it holds when we expect over w, with probability one. Q.E.D.

Finally, consider the existence of a best policy among those policies $d \neq 0$. These policies involve taking at least one sample x_1. Having observed x_1, D will have a revised distribution F^{x_1} and a new problem formally identical to the original. If, with probability one, there exists an optimal policy for the new problem, then we have our result. But the sufficient conditions of Theorems A.4 and A.5 carry over, with probability one, to the new problem.

Appendix B

We supply the details here for the proof of the existence of y^+ as explained in Lemma 3. For any policy $d \not\equiv 0$ we have the identity

$$E_F \text{ (value of following } d) \equiv E_{F^1}\{(\text{payoff on first draw}) + E_{F^{x_1}} (\text{value of following } d \mid \text{outcome of first draw})\}. \tag{B.1}$$

Here F is the current joint distribution over all sequences of future draws, F^1 is the marginal distribution derived from F over the first draw x_1, and F^{x_1} is the *joint* distribution (over all sequences of draws $(x_2, x_3, \ldots)$) as revised in view of the outcome of the first draw. Rewriting (B.1) in our notation, we obtain

$$W(y, F, d) \equiv \frac{1}{1+r} E_{F^1} k_1(x_1) + \frac{1}{1+r} \int W(\max(y, u(x_1)), F^{x_1}, d')\, dF^1(x_1) \tag{B.2}$$

where d' is the continuation of policy d for draws subsequent to the first.

From (B.2) we have

$$W(y, F, d) - \frac{y}{r} = \frac{1}{1+r} [E_{F^1} k(x_1) - y] + \frac{1}{1+r} \int \left[W(\max(u, u(x_1)), F^{x_1}, d') - \frac{y}{r}\right] dF^1(x_1). \tag{B.3}$$

The first term falls with y at the rate $1/(1+r)$. Rewriting (7) of Section 2, we obtain

$$W(y', F, d') - \frac{y'}{r} < W(y, F, d') - \frac{y}{r} \tag{B.4}$$

for $y' > y$ and any d' and F. It follows from (B.4) that the integrand in the second term of (B.3) is nonincreasing in y. Thus, $W(y, F, d) - y/r$ falls at least at the rate $1/(1+r)$. In other words, if y_0 is some arbitrary level of best available utility, then the function $W(y, F, d)$ is bounded above by the linear function

$$L(y; d) \equiv W(y_0, F, d) + (y - y_0)/(r(1 + r)) \qquad \text{(B.5)}$$

for $y \geqslant y_0$ (since $L(y; d) - y/r$ falls exactly at the rate $1/(1 + r)$ and $L(y; d)$ equals $W(y_0, F, d)$ at y_0). Because $V(y_0, F)$ is greater than or equal to $W(y_0, F, d)$ for *all* $d \neq 0$, we have

$$\begin{aligned} L(y) &\equiv V(y_0, F) + \frac{y - y_0}{r(1 + r)} \\ &\geqslant W(y_0, F, d) + \frac{y - y_0}{r(1 + r)} \geqslant W(y, F, d) \qquad \text{(B.6)} \end{aligned}$$

for $y \geqslant y_0$. Since y/r and $L(y)$ are linear in y with y/r being steeper, there exists a $y^+ \geqslant y_0$ such that $y^+/r > L(y^+)$. Hence, by (B.6), $y^+/r > W(y^+, F, d)$ for all $d \neq 0$, and therefore the policy "stop" ($d = 0$) is optimal at y^+.

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